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Supersymmetry and algebraic Darboux transformations

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Abstract

We describe a class of algebraically solvable SUSY models by considering the deformation of invariant polynomial flags by means of the Darboux transformation. The algebraic deformations corresponding to the addition of a bound state to a shape-invariant potential are particularly interesting. The polynomial flags in question are indexed by a deformation parameter $m = 1, 2, \ldots$, and lead to new algebraically solvable models. We illustrate these ideas by considering deformations of the hyperbolic Pöschl–Teller potential.

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1. Introduction

Our purpose in this paper is to show how new classes of exactly solvable supersymmetric quantum mechanical Hamiltonians arise in a natural fashion from the application of the Darboux transformation to classes of second-order linear differential operators that preserve flags of vector spaces generated by univariate polynomials. These new Hamiltonians and their bound states have closed analytic expressions in terms of elementary functions, and their qualitative behaviour is both natural and significant from a physical point of view.

The classical approach to the Darboux transformation is based on a formal eigenfunction ϕ of a Schrödinger operator H, which is used to factorize H as a product of first-order operators. Depending on whether ϕ, ϕ^{-1}, or neither are square integrable, one obtains state-deleting, state-adding, or isospectral[4] Darboux transformations [1–3], in which the supersymmetric partner Hamiltonian is obtained by reversing the order of these factors. The principle of

[4] The *state-deleting, state-adding and isospectral* transformations are named, respectively, cases a, b, c in [1] and cases T_ψ, T_n, T_l/T_r in [4].

our approach is to consider only those factorizations for which the effect of the Darboux transformation on functions is to map polynomials to polynomials. These are given a simple characterization in our paper in terms of the starting formal eigenfunction. We will refer to this transformation as the *algebraic Darboux transformation*. The new Hamiltonians obtained in this fashion will be exactly solvable in the precise algebraic sense that they will also admit complete invariant flags of polynomial subspaces.

When a parametrized family of potentials is closed with respect to the state-deleting Darboux transformation it is said to be *shape-invariant* [5, 6], and in this case the iteration of the transformation furnishes a complete description of the spectrum and eigenfunctions[5]. For the shape-invariant potentials, the underlying invariant flag of polynomials is the full polynomial module.

Thus, to obtain deformations of a shape-invariant potential one must consider the two-parameter family of *state-adding Darboux transformations*. These were first applied to the harmonic oscillator in [9]. The general theory was developed in [2, 1] and connections to the inverse scattering method noted in [7]. However, as noted in [9, 10], the general form of the deformed potential can only be expressed by a formal power series, or as the integral of eigenfunctions of the original Hamiltonian—in contrast to the original potential, which is an elementary function with bound states also described by elementary functions.

Our main emphasis in this paper is to obtain examples of exactly solvable potentials which lie outside the shape-invariant class, which can be expressed in closed analytic form, and which have qualitative properties that make them relevant to the description of physically realistic situations. These are obtained by the application of special instances of the state-adding Darboux transformation to shape invariant potentials. There is a countable infinity of algebraic state-adding transformations of a shape-invariant potential [3], indexed by an integer m. We will show the precise manner in which the mth algebraic state-adding transformation deforms the invariant polynomial flag of a shape-invariant potential, and we calculate the explicit basis of the deformed flags for the cases $m = 1$ and $m = 2$. As an illustrative example, we discuss in detail the algebraic deformation of the hyperbolic Pöschl–Teller potential.

2. Darboux transformations

2.1. The self-adjoint case

Consider the Schrödinger operator

$$H = -D_{xx} + u, \tag{1}$$

where $u(x)$, $x \in \mathbb{R}$ is continuous, real-valued and bounded from below[6]. Consequently, the restriction of H to a certain dense subspace $\mathcal{D}(H) \subset L^2(\mathbb{R})$ is a self-adjoint operator. Consider a formal eigenfunction $\phi > 0$ of the following eigenvalue equation:

$$H\phi = \lambda_0\phi.$$

The key idea of the supersymmetric or Darboux transformation is the fact that to every ϕ there corresponds a factorization of H as

$$H - \lambda_0 = A^\dagger A, \tag{2}$$

where

$$A = D_x - (\log\phi)_x, \qquad A^\dagger = -D_x - (\log\phi)_x. \tag{3}$$

[5] Note that Infeld and Hull's contribution [8] is at the root of many of the subsequent developments in the subject.
[6] In this paper, we restrict to potentials whose domain is the entire real line. Radial potentials can be treated in an analogous manner.

We shall refer to ϕ as the *factorization function*, and to λ_0 as the *factorization energy*. The supersymmetric partner potential is the operator defined by the commutation of the factors

$$\hat{H} = -D_{xx} + \hat{u} = AA^{\dagger} + \lambda_0, \qquad \hat{u} = u - 2(\log\phi)_{xx}. \tag{4}$$

The transformed potential $\hat{u}$ is continuous since ϕ vanishes nowhere. In this way, $\hat{H}$ is self-adjoint and semi-bounded on some dense domain $\mathcal{D}(\hat{H})$. The operators H and $\hat{H}$ satisfy the intertwining relation[7]

$$AH = \hat{H}A, \tag{5}$$

which implies the following relation between the eigenfunctions of the two operators:

$$H\psi = \lambda\psi, \qquad \hat{H}\hat{\psi} = \lambda\hat{\psi}, \qquad \hat{\psi} = A\psi. \tag{6}$$

The spectral properties of this transformation are governed by one of the following three possibilities [1, 4, 14], see also [15].

(i) *State-deleting transformation:* ϕ is square integrable (and since it is nodeless, it must be the ground-state wavefunction of H). The operator A maps $\mathcal{D}(H)$ onto $\mathcal{D}(\hat{H})$, with a one-dimensional kernel generated by ϕ. The nth bound state of H is mapped to the $(n-1)$th bound state of $\hat{H}$. Correspondingly, the transformed spectrum differs from the spectrum of H by the removal of λ_0, the lowest eigenvalue.

(ii) *State-adding transformation:* ϕ^{-1} is square integrable. The operator A maps the nth bound state of H to the $(n+1)$th bound state of $\hat{H}$. It is one-to-one on $\mathcal{D}(H)$, but not onto $\mathcal{D}(\hat{H})$; the new ground state is not in the image of $A : \mathcal{D}(H) \to \mathcal{D}(\hat{H})$. The spectrum of $\hat{H}$ differs from that of H by the addition of a lowest eigenvalue, namely λ, with the ground state given by ϕ^{-1}. There is a two-parameter family of state-adding transformations labelled by the energy and shape parameter (i.e., a one-parameter family exists for every λ strictly smaller than the infimum of the spectrum of H).

(iii) *Isospectral transformation:* neither ϕ nor ϕ^{-1} are square integrable. The operator A defines a linear isomorphism from $\mathcal{D}(H)$ to $\mathcal{D}(\hat{H})$. It transforms the nth bound state of H to the nth bound state of $\hat{H}$. Two isospectral Darboux transformations exist for every λ strictly smaller than the infimum of the spectrum of H.

2.2. *The general form and covariance*

In this paragraph we will consider Darboux transformations of an arbitrary second-order operator, the general form of which is

$$T = pD_{zz} + qD_z + r, \tag{7}$$

where we assume that $p(z) < 0$ on the domain of interest. The above operator is related to a Schrödinger operator (1) by the change of variables

$$x = \int (-p)^{-\frac{1}{2}}\,\mathrm{d}z, \tag{8}$$

and gauge transformation

$$H = H_T = \mathrm{e}^{\rho} T\,\mathrm{e}^{-\rho} = -D_{xx} + u_T, \tag{9}$$

where

$$\rho = \int \frac{1}{2}p^{-1}\left(q - \frac{1}{2}p_z\right)\mathrm{d}z.$$

[7] We could have started with the intertwining relation (5), where the operators are as in (1), (3) and (4), obtaining the factorization as a result [11]. This last approach admits the generalization to higher order intertwining operators, see, e.g., [12, 13] and references therein.

The potential is given by

$$u = u_T = \tfrac{1}{4}p_{zz} - \tfrac{1}{2}q_z - \tfrac{1}{4}p^{-1}\big(q - \tfrac{1}{2}p_z\big)\big(q - \tfrac{3}{2}p_z\big) + r. \tag{10}$$

Since gauge transformations and changes of variable are homomorphisms of the ring of differential operators, the Darboux transformation is covariant with respect to these operations.

Although it is customary to work in the Schrödinger gauge as in section 2.1, the Darboux transformation can be defined relative to a general coordinate and an arbitrary choice of gauge, as shown below. Indeed, let

$$T\phi = \lambda_0\phi, \tag{11}$$

be a factorization eigenfunction. For every a and b such that

$$(\log\phi)_z = a/b, \tag{12}$$

we obtain the following factorization of T:

$$T = BA + \lambda_0, \tag{13}$$

where

$$A = bD_z - a = b(D_z - (\log\phi)_z), \tag{14}$$

$$B = \frac{p}{b}\left(D_z + \frac{a - b_z}{b} + \frac{q}{p}\right) = (pD_z + p(\log\phi)_z + q)b^{-1}. \tag{15}$$

We can then define the partner operator

$$\hat{T} = AB + \lambda_0, \tag{16}$$

and observe the following intertwining relations:

$$\hat{T}A = AT, \qquad B\hat{T} = TB. \tag{17}$$

The choice of an operator $\hat{T}$ from a given T is not unique, but rather covariant with respect to the choice of a and b in (12). A different choice, say,

$$(\log\phi)_z = a'/b',$$

will lead to a different partner operator $\hat{T}'$, which differs from $\hat{T}$ by a gauge transformation. To wit,

$$b^{-1}\hat{T}b - \lambda_0 = (b')^{-1}\hat{T}'b' - \lambda_0 = (D_z - (\log\phi)_z)(pD_z + q + p(\log\phi)_z),$$

and hence,

$$\hat{T}' = \left(\frac{b'}{b}\right)\hat{T}\left(\frac{b'}{b}\right)^{-1}.$$

Therefore, $H_{\hat{T}} = H_{\hat{T}'}$, using the notation introduced in (9).

In particular, when $q = 1/2p_z$, i.e., when the operator is in the self-adjoint gauge, we may take

$$b = (-p)^{1/2}, \qquad a = (-p)^{1/2}(\log\phi)_z,$$

and obtain $B = A^\dagger$. Consequently, each of the possible factorizations (13) is equivalent, after a change of variables and a gauge transformation, to the self-adjoint factorization (2).

To perform an inverse transformation, we use

$$\hat{\phi} = b\exp\left(-\int\left(\frac{q}{p} + \frac{a}{b}\right)\mathrm{d}z\right) \tag{18}$$

as the factorization function. A simple calculation shows that

$$\hat{T}\hat{\phi} = \lambda_0\hat{\phi}, \tag{19}$$

and $\hat{T}$ has the following form:

$$\hat{T} = pD_{zz} + \hat{q}D_z + \hat{r},$$

where

$$\hat{q} = q + p_z - 2p(\log b)_z, \tag{20}$$

$$\hat{r} = -p(\log b)_{zz} + p(\log b)_z^2 - (p_z + q)(\log b)_z - \frac{2pa^2}{b^2} + (p_z - 2q)\frac{a}{b} + q_z + 2\lambda_0 - r. \tag{21}$$

A particularly interesting factorization is obtained by taking

$$\hat{a} = p(b_z - a) - qb, \qquad \hat{b} = pb. \tag{22}$$

Thus, $(\log\hat{\phi})_z = \hat{a}/\hat{b}$, and from (14) and (15) we obtain

$$\hat{T} = \hat{B}\hat{A} + \lambda_0,$$

where

$$\hat{A} = \hat{b}D_z - \hat{a} = bp\left(D_z + \frac{a - b_z}{b} + \frac{q}{p}\right) = b^2B, \tag{23}$$

$$\begin{aligned}\hat{B} &= \frac{p}{\hat{b}}\left(D_z + \frac{\hat{a} - \hat{b}_z}{\hat{b}} + \frac{\hat{q}}{p}\right)\\ &= \frac{1}{b}\left(D_z - \frac{a}{b} - \frac{2b_z}{b}\right) = b\left(D_z - \frac{a}{b}\right)b^{-2} = Ab^{-2}.\end{aligned} \tag{24}$$

It follows that

$$\hat{\hat{T}} = \hat{A}\hat{B} + \lambda_0 = b^2Tb^{-2}.$$

Thus, the factorization transformation (23) is a quasi-inverse to the factorization transformation (14), i.e., it is an inverse modulo gauge transformations. An exact inverse could be obtained by choosing

$$\hat{a}' = \frac{p}{b^2}(b_z - a) - \frac{q}{b}, \qquad \hat{b}' = \frac{p}{b}.$$

However, (22) will be important when we consider algebraic factorizations.

2.3. *Algebraic factorizations*

We will say that a second-order differential operator is *exactly solvable by polynomials* (P.E.S.) if it is equivalent, by a change of variable and a gauge transformation, to a second-order operator T that preserves an infinite flag of finite-dimensional *polynomial* subspaces

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M} = \cup_n\mathcal{M}_n, \qquad T\mathcal{M}_i \subseteq \mathcal{M}_i. \tag{25}$$

As part of this definition we include the following assumptions:

(E1) We assume that each $\mathcal{M}_n$ is an n-dimensional subspace of

$$\mathcal{P}_{n+m-1} = \langle 1, z, z^2, \ldots, z^{n+m-1}\rangle,$$

and thus the co-dimension of $\mathcal{M}_n$ in $\mathcal{P}_{n+m-1}$ is m.

(E2) There is no spectral degeneracy. The action of T is upper-triangular relative to a basis adapted to the above flag, and hence possesses an infinite list of eigenpolynomials. We assume that the corresponding eigenvalues are distinct.

This definition is similar to the definition of *exact solvability* introduced in [16]. Of particular interest is the subclass of P.E.S. operators which admit a gauge such $\mathcal{M}_1 = \mathbb{R}$. If this condition holds, we will say that the operator satisfies the *algebraic ground-state condition*.

We will now consider the following question. Supposing that T is a P.E.S. operator with invariant flag (25), what are the conditions on a factorization function ϕ such that the partner operator $\hat{T}$ is also P.E.S.? In this regard, we will say that a factorization eigenfunction (11) is of *algebraic type* whenever ϕ_z/ϕ is a rational function, which is equivalent to the condition that A transforms polynomials into polynomials. We now define three kinds of algebraic Darboux transformation in analogy to the three cases discussed in section 2.1. Throughout, ϕ is a factorization function of algebraic type, $\mathcal{M}_n$ refers to the invariant flag of T, while $\hat{\mathcal{M}}_n$ refers to the invariant flag of $\hat{T}$.

(A1) We speak of an *algebraic state-deleting transformation,* whenever $\mathcal{M}_1 = \langle\phi\rangle$ and $\hat{\mathcal{M}}_n = A\mathcal{M}_{n+1}$. Note that in this case, the operator A annihilates $\mathcal{M}_1$. Also note that we include here the assumption that ϕ is a polynomial.

(A2) We speak of an *algebraic state-adding transformation* whenever $\hat{\phi}$, as given in (18), is a rational function, say $\hat{\phi} = \tilde{a}/\tilde{b}$, and when $\hat{\mathcal{M}}_1 = \langle\tilde{a}\rangle$, $\hat{\mathcal{M}}_{n+1} = \tilde{b}A\mathcal{M}_n \oplus \langle\tilde{a}\rangle$. The polynomials $\tilde{a} = \tilde{a}(z)$ and $\tilde{b} = \tilde{b}(z)$ must satisfy

$$\frac{b_z}{b} - \frac{q}{p} - \frac{a}{b} = \frac{\tilde{a}_z}{\tilde{a}} - \frac{\tilde{b}_z}{\tilde{b}} = \left(\log\frac{\tilde{a}}{\tilde{b}}\right)_z. \tag{26}$$

(A3) We speak of an *algebraically isospectral transformation,* whenever $\hat{\mathcal{M}}_n = A\mathcal{M}_n$.

We should point out that exact solvability by polynomials, just like quasi-exact solvability, is a formal concept. The requirement of square integrability of the algebraic sector in the physical coordinate and gauge is an extra constraint, which needs to be analysed separately, cf [17].

Of particular interest is the case where (A2) holds, and where, in addition, both T and $\hat{T}$ satisfy the algebraic ground-state condition.

Proposition 1. *Let T, ϕ be as in (11) and (12), with $a = a(z), b = b(z)$ relatively prime polynomials, and with $\hat{T}, \hat{\phi}$ as in (16), (18) and (19). Suppose that T satisfies the algebraic ground-state condition. The following are equivalent:*

(i) $T \mapsto \hat{T}$ *is an algebraic state-adding transformation with* $\hat{T}$ *satisfying the algebraic ground-state condition.*
(ii) $pa_z + (r - \lambda_0)b = 0$.
(iii) $\hat{\phi}$ *is a constant.*

Proof. Since T satisfies the algebraic ground-state condition, the r in (7) is, without loss of generality, a constant. The implication (iii) ⇒ (i) follows directly from (A2).

Let us now prove the converse. Suppose that (i) holds. By (15) and (18) we have

$$B = \frac{p}{b}\left(D_z - \frac{\hat{\phi}_z}{\hat{\phi}}\right) = \left(\frac{p\hat{\phi}}{b}D_z\right)\hat{\phi}^{-1}.$$

Hence,

$$\begin{aligned} T = BA + \lambda_0 &= \left(\frac{p\hat{\phi}}{b}D_z\right)\left(\frac{b}{\hat{\phi}}D_z - \frac{a}{\hat{\phi}}\right) + \lambda_0 \\ &= pD_{zz} + qD_z - \frac{p\hat{\phi}}{b}\left(\frac{a}{\hat{\phi}}\right)_z + \lambda_0. \end{aligned} \tag{27}$$

By (14), $A(1) = -a$. By (A2), $\hat{\phi} = \tilde{a}/\tilde{b}$ is a rational function with

$$\hat{\mathcal{M}}_1 = \langle \tilde{a} \rangle, \qquad \hat{\mathcal{M}}_2 = \langle \tilde{a}, a\tilde{b} \rangle.$$

Since $\hat{T}$ satisfies the algebraic ground-state condition,

$$\frac{a\tilde{b}}{\tilde{a}} = \frac{a}{\hat{\phi}}$$

is a polynomial. Hence, by (7) and (27),

$$\left(\frac{a}{\hat{\phi}}\right)_z = (\lambda_0 - r)\frac{b}{p\hat{\phi}}.$$

By (E2), $\lambda_0 \neq r$, and hence $\hat{\phi}$ divides b, as well. Therefore, $\hat{\phi}$ must be a constant.

Finally, let us show (iii) $\Leftrightarrow$ (ii). By (18), (iii) is true if and only if

$$\frac{b_z - a}{b} - \frac{q}{p} = 0. \tag{28}$$

By (11) and (12),

$$p\left(\frac{a_z}{b} - \frac{a}{b}\frac{b_z}{b}\right) + \left(\frac{a}{b}\right)^2 + \frac{qa}{b} + r - \lambda_0 = 0,$$

or equivalently,

$$pa_z + (r - \lambda_0)b = pa\left(\frac{b_z - a}{b} - \frac{q}{p}\right).$$

Therefore, (ii) holds if and only if (28) does. $\square$

3. Algebraic deformations of shape-invariant potentials

3.1. Shape invariance

Let us recall that a parametrized potential is commonly called *shape-invariant* [5, 6] if the state-deleting Darboux transformation preserves the form of the potential while altering the value of the parameters. In the preceding section, we pointed out that the Darboux transformation is covariant with respect to gauge transformations and changes of variable. As a consequence, the notion of shape invariance makes perfect sense for general second-order operators, and not just for operators in Schrödinger form. Thus, we will adapt the usual definition and say that a parametrized family of P.E.S. operators is shape-invariant if that family is closed with respect to the algebraic, state-deleting (A1) Darboux transformation.

We now describe an important class of shape-invariant, P.E.S. operators. We define the *standard polynomial flag* to be

$$\mathbb{R} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots, \qquad \mathcal{P}_n = \langle 1, z, \ldots, z^n \rangle. \tag{29}$$

The general form of a second-order operator T that preserves the standard flag is

$$T = pD_{zz} + qD_z + r, \tag{30}$$

where $p = p(z)$ and $q = q(z)$ are, respectively, second- and first-degree polynomials, and where r is a constant. The family of operators described by (30) is shape-invariant in the above sense. The ground state is given by $\phi = 1$ with $\lambda_0 = r$, and the factorization is simply

$$T = (pD_z + q)D_z + \lambda_0.$$

The partner operator

$$\hat{T} = D_z(pD_z + q) + \lambda_0 = pD_{zz} + \hat{q}D_z + \hat{r},$$

Table 1. Shape-invariant potentials on the line.

	$\mathrm{I_e}$	$\mathrm{I_o}$	II	$\mathrm{III_e}$	$\mathrm{III_o}$
p	$-4z$	$-4z$	$-z^2$	$z(1-z)$	$z(1-z)$
q	$4z-2$	$4z-6$	$(2A-1)z-1$	$\left(A-\frac{3}{2}\right)z+1-A$	$\left(A-\frac{5}{2}\right)z+1-A$
r	1	3	$-A^2$	$-\left(\frac{A}{2}-\frac{1}{4}\right)^2$	$-\left(\frac{A}{2}-\frac{3}{4}\right)^2$
e^ρ	$\mathrm{e}^{-\frac{x^2}{2}}$	$x\,\mathrm{e}^{-\frac{x^2}{2}}$	$\exp\left(-\frac{1}{2}\mathrm{e}^{-x}-Ax\right)$	$\cosh\left(\frac{x}{2}\right)^{-A+\frac{1}{2}}$	$\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)^{-A+\frac{1}{2}}$
$z(x)$	x^2	x^2	e^x	$\cosh^2\left(\frac{x}{2}\right)$	$\cosh^2\left(\frac{x}{2}\right)$
U	x^2	x^2	$\frac{1}{4}\mathrm{e}^{-2x}-\left(A+\frac{1}{2}\right)\mathrm{e}^{-x}$	$\frac{1}{4}\left(\frac{1}{4}-A^2\right)\mathrm{sech}^2\left(\frac{x}{2}\right)$	$\frac{1}{4}\left(\frac{1}{4}-A^2\right)\mathrm{sech}^2\left(\frac{x}{2}\right)$

retains the form (30), with

$$\hat{q} = p_z + q, \qquad \hat{r} = q_z + r.$$

The corresponding non-singular potential forms—see (8)–(10) for the transformation formulae and [3] for their derivation—are shown in table 1. With the assumptions taken in this paper, these are the well-known shape-invariant potential families: the harmonic oscillator (I), the Morse potential (II) and the hyperbolic Pöschl–Teller potentials (III). Since potentials (I) and (III) are even functions, the corresponding eigenfunctions have a well-defined parity. Consequently, these potentials possess two algebraic sectors, i.e., they are exactly solvable by polynomials in two distinct ways (see [16, 17] for a discussion of the notion of algebraic sector and [18] for an algebraic explanation of potentials with multiple algebraic sectors). The even sector corresponds to an even gauge factor, and the odd sector to an odd gauge factor. The parity of the algebraic sectors is reversed by a Darboux transformation.

3.2. Deformations of the standard flag

Let $a = a(z), b = b(z)$ be relatively prime polynomials, and let $g = g(z)$ be a polynomial that divides a_z, b and $b_z - a$. We assume that $g(z)$ has no zeros in the interval of interest. Consider the differential operators

$$B = g^{-1}D_z, \qquad A = bD_z - a, \tag{31}$$

and note that, by assumption, the second-order operator

$$T = BA = pD_{zz} + qD_z - a_z g^{-1} \tag{32}$$

has polynomial coefficients $p = bg^{-1}, q = (b_z - a)g^{-1}$. We will say that A, B constitute a *deformation pair* of order $m = \deg(g)$ if T leaves invariant $\mathcal{M}_n = \mathcal{P}_{n-1}$ for all n, i.e., if T leaves invariant the standard polynomial flag. Deformation pairs are of interest because they provide non-trivial examples of exactly solvable Hamiltonians outside the shape-invariant class. As usual, we define a partner operator

$$\hat{T} = AB = pD_{zz} + \hat{q}D_z, \qquad \hat{q} = -(bg^{-1}g_z + a)g^{-1}, \tag{33}$$

and a partner flag

$$\mathbb{R} = \hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}}_2 \subset \hat{\mathcal{M}}_3 \subset \cdots, \qquad \hat{\mathcal{M}}_n = A\mathcal{P}_{n-2} \oplus \mathbb{R},$$

which we will refer to as a *deformation* of the standard polynomial flag.

Proposition 2. *Every $\hat{\mathcal{M}}_n$ is a codimension m subspace of $\mathcal{P}_{n+m-1}$.*

Proof. By assumptions on the operators T and $\hat{T}$, we have that $a_z g^{-1}$ is constant, and that $\deg(bg^{-1}) \leqslant 2$. This implies that $\deg(a) = m + 1$ and that $\deg(b) \leqslant m + 2$. Therefore, $\hat{\mathcal{M}}_n \subset \mathcal{P}_{n+m-1}$. Since a, b are relatively prime, A does not annihilate any polynomial, and hence $\dim \hat{\mathcal{M}}_n = n$. □

Proposition 3. *The partner operator $\hat{T}$ is P.E.S.*

Proof. The operator $\hat{T}$ preserves the deformed flag because of the intertwining relation

$$\hat{T}A = AT,$$

and because $\hat{T}$ annihilates $\mathbb{R} = \hat{\mathcal{M}}_1$ by construction. Condition (E1) is true by the preceding proposition. We noted above that A and hence that T does not annihilate any polynomial. Hence 0 is not an eigenvalue of T, which proves condition (E2). □

Let us also note that a deformation pair satisfies the conditions of proposition 1, and in particular $T \mapsto \hat{T}$ is of type (A2).

Proposition 4. *The deformed subspaces $\hat{\mathcal{M}}_n$ can be characterized as the subspace of $\mathcal{P}_{n+m-1}$ consisting of all polynomials $f = f(z)$ such that g divides f_z.*

Proof. First, note that for

$$f = Ah, \qquad h \in \mathcal{M}_{n-1} = \mathcal{P}_{n-2}, \qquad f \in \hat{\mathcal{M}}_n$$

we have that $g^{-1} f_z = Th$, which proves that f_z is divisible by g. In order to prove the converse, let us note that the subspace of all $f \in \mathcal{P}_{n+m-1}$ such that f_z is divisible by g is n-dimensional. However, $\dim \hat{\mathcal{M}}_n = n$ by proposition 2, which proves the claim. □

We will now show an explicit basis of $\hat{\mathcal{M}}_n$ in the cases $m = 1$ and $m = 2$. In the first instance, since the subspaces $\mathcal{P}_n$ are invariant with respect to translations, we may without loss of generality assume that $g(z) = z$. By (32) necessarily, $b = pz$, where $p = p_2 z^2 + p_1 z + p_0$ is a polynomial of degree 2 or less. In order for g to divide a_z and $b_z - a$ we must have

$$A = (p_2 z^2 + p_1 z + p_0) z D_z - (a_2 z^2 + p_0), \tag{34}$$

where a_2, p_0, p_1, p_2 are arbitrary real numbers. Let us also assume that $a_2 \neq 0$ and that a_2/p_2 is not a positive integer. If these generic conditions hold, then the subspaces of the partner flag are given by

$$A\mathcal{P}_{n-2} \oplus \mathbb{R} = \hat{\mathcal{M}}_n = \langle 1, z^2, z^3, \ldots, z^n \rangle.$$

The above monomial-generated subspace is exceptional in that it admits a seven-dimensional vector space of second-order operators that preserve it [19], and consequently can be used to construct novel instances of exactly solvable and quasi-exactly solvable operators [3, 18].

Turning to the case $m = 2$, we limit our discussion to the generic case of $g(z)$ with distinct roots. By scaling and translating z, as necessary, we may assume, without loss of generality, that $g = z^2 - 1$. By (32), in order for g to divide a_z and $b_z - a$ we must have

$$A = (p_2 z^2 + p_1 z + p_0)(z^2 - 1) D_z + (p_2 + p_0)(z^3 - 3z) - 2p_1, \tag{35}$$

where p_0, p_1, p_2 are arbitrary real numbers. Let us also assume that $p_2 + p_0 \neq 0$ and that $-p_0/p_2$ is not a positive integer. If these generic conditions hold, then the subspaces of the partner flag are given by

$$A\mathcal{P}_{n-2} \oplus \mathbb{R} = \hat{\mathcal{M}}_n = \langle 1, \pi_3(z), \pi_4(z), \ldots, \pi_{n+1}(z) \rangle,$$

where

$$\pi_{2k+1}(z) = z^{2k+1} - (2k+1)z, \qquad \pi_{2k}(z) = z^{2k} - kz^2. \tag{36}$$

The above polynomials $\pi = \pi(z)$ have the property that π_z is divisible by $z^2 - 1$. The resulting polynomial subspaces $\mathcal{M}_n$ are preserved by the following second-order operators:

$$T_3 = z^3 D_{zz} + \left((1-n)z^2 - 5 + n - \frac{4}{z^2-1}\right) D_z,$$
$$T_2 = (z^2-1)D_{zz} - 2zD_z,$$
$$T_1 = zD_{zz} - 2\left(1 + \frac{2}{z^2-1}\right) D_z,$$
$$T_0 = D_{zz} + \left(z - \frac{4z}{z^2-1}\right) D_z.$$

3.3. *Algebraic deformations of the hyperbolic Pöschl–Teller potential*

The hyperbolic Pöschl–Teller potential [21], which includes the class of reflectionless 1-soliton potentials [23], has the form

$$U_{\mathrm{PT}}(x) = \frac{1}{4}\left(\frac{1}{4} - \alpha^2\right) \operatorname{sech}^2\left(\frac{x}{2}\right). \tag{37}$$

The general solution [24, sections 2.9] of the corresponding Schrödinger equation

$$H_{\mathrm{PT}}(\phi) = -\phi_{xx} + U_{\mathrm{PT}}\phi = -k^2\phi$$

can be given as

$$\phi_{\mathrm{PT}}(x; k, C_0, C_1) = \cosh\left(\frac{x}{2}\right)^{\frac{1}{2}-\alpha} \left\{ C_0\, {}_2F_1\left(-\frac{\alpha}{2} + \frac{1}{4} + k, -\frac{\alpha}{2} + \frac{1}{4} - k, \frac{1}{2}; -\sinh^2\left(\frac{x}{2}\right)\right) \right.$$
$$\left. + C_1 \sinh\left(\frac{x}{2}\right) {}_2F_1\left(-\frac{\alpha}{2} + \frac{3}{4} + k, -\frac{\alpha}{2} + \frac{3}{4} - k, \frac{3}{2}; -\sinh^2\left(\frac{x}{2}\right)\right) \right\},$$

where ${}_2F_1(a, b, c; z)$ also denotes the analytic continuation of the hypergeometric function to $\mathrm{Re}(z) < 0$. For $\alpha > 1/2$, the potential (37) has $\lceil \alpha - \frac{1}{2} \rceil$ bound states

$$\psi_{\mathrm{pt},i}(x), \qquad 0 \leqslant i < \alpha - \tfrac{1}{2}.$$

The even bound states are given by [3]

$$\psi_{\mathrm{pt},2j}(x) \propto \phi_{\mathrm{PT}}\left(x; \frac{\alpha}{2} - j - \frac{1}{4}, 1, 0\right)$$
$$\propto \cosh\left(\frac{x}{2}\right)^{\frac{1}{2}-\alpha} P_j^{(-\frac{1}{2},-\alpha)}(\cosh x). \tag{38}$$

The odd ones are given by

$$\psi_{\mathrm{pt},2j+1}(x) \propto \phi_{\mathrm{PT}}\left(x; \frac{\alpha}{2} - j - \frac{3}{4}, 0, 1\right)$$
$$\propto \sinh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right)^{\frac{1}{2}-\alpha} P_j^{(\frac{1}{2},-\alpha)}(\cosh x), \tag{39}$$

where $P_j^{(a,b)}(z)$ are the Jacobi polynomials. We focus on deformations of potentials with bound states only, i.e., we must take $\alpha > \frac{1}{2}$. In order to have a well-defined state-adding Darboux transform of the hyperbolic Pöschl–Teller potential, we must consider the solutions

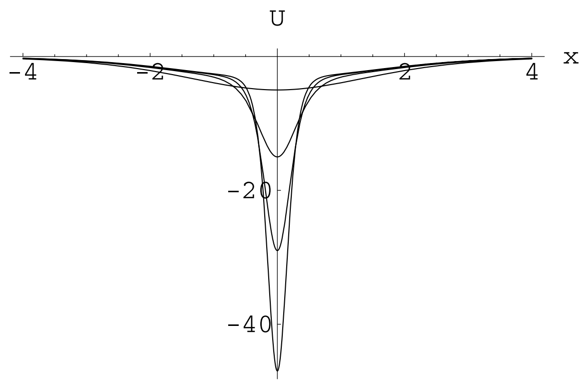

Figure 1. Algebraic deformations $U_{\mathrm{PT}}^{(m)}(x)$ of the hyperbolic Pöschl–Teller potential (41) with $\alpha = 4$ and $m = 0, 1, 2$ and 3.

ϕ_{PT} which correspond to an energy below the spectral minimum and are nowhere vanishing. Since the spectral minimum is $-\left(\frac{1}{2}-\alpha\right)^2$ we take $|k| > \frac{1}{2}-\alpha$. Now, the two-parameter family of state-adding Darboux transformations is given by the transformation functions:

$$\phi_{\mathrm{PT}}(x; k, 1, t), \qquad |t| \leqslant 2\frac{\Gamma\left(\frac{3}{4}+k-\frac{\alpha}{2}\right)\Gamma\left(\frac{3}{4}+k+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1}{4}+k-\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{4}+k+\frac{\alpha}{2}\right)}, \tag{40}$$

with the extreme values of the shape parameter t corresponding to an isospectral transformation [3]. Supersymmetric partners of the Pöschl–Teller potential have been previously considered in [10, 22]. In general, for an arbitrary value of the energy parameter k the partner potential will be defined formally by a power series. However, for specific values of k the log derivative of ϕ_{PT} will be a polynomial in $\cosh x$ thus giving rise to an algebraic deformation. It can be shown that these algebraic deformations occur precisely for the following countable subset of (40):

$$\begin{aligned}\phi_{\mathrm{PT}}^{(m)}(x) &= \phi_{\mathrm{PT}}\left(x, -\frac{\alpha}{2}-\frac{1}{4}-m, 1, 0\right)\\ &\propto \cosh\left(\frac{x}{2}\right)^{\frac{1}{2}+\alpha} P_m^{(-\frac{1}{2},\alpha)}(\cosh(x)),\end{aligned}$$

The resulting deformed potentials, as given by (4), have the form

$$U_{\mathrm{PT}}^{(m)}(x) = -\frac{1}{4}\left(\alpha+\frac{1}{2}\right)\left(\alpha+\frac{3}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right) - 2\left(\log P_m^{(-\frac{1}{2},\alpha)}(\cosh x)\right)_{xx}, \tag{41}$$

and have been plotted in figure 1.

More specifically, the first ($m = 1$) and second ($m = 2$) deformations have the following forms:

$$U_{\mathrm{PT}}^{(1)}(x) = U_{\mathrm{PT}}^{(0)}(x) + \frac{2\alpha+1}{z_1} - \frac{4(\alpha+1)}{z_1^2},$$

where

$$z_1 = \tfrac{1}{2}((2\alpha+3)\cosh x - (2\alpha+1)),$$

and

$$U_{\mathrm{PT}}^{(2)}(x) = U_{\mathrm{PT}}^{(0)}(x) + \frac{(2\alpha+1)\left(\beta\left(z_2^3+3z_2\right)-2z_2^2-2\right)-8}{\left(z_2^2-1\right)^2},$$

where

$$z_2 = \frac{1}{4}\beta((2\alpha+7)\cosh x - (2\alpha+1)), \qquad \beta = \sqrt{\frac{2\alpha+5}{3\alpha+6}}.$$

The algebraic state-adding Darboux transformation corresponds to the first-order operator

$$\begin{aligned} A_{\mathrm{PT}}^{(m)} &= D_x - \left(\log\phi_{\mathrm{PT}}^{(m)}\right)_x \\ &= D_x - \frac{1}{2}\left(m+\alpha+\frac{1}{2}\right)\sinh x\,\frac{P_{m-1}^{(\frac{1}{2},\alpha+1)}(\cosh x)}{P_m^{(-\frac{1}{2},\alpha)}(\cosh x)}. \end{aligned}$$

Both the undeformed and the deformed potentials are even functions, and consequently the corresponding Hamiltonians leave invariant the spaces of odd and even functions. The Darboux transformation changes parity in this case. In particular, the deformed even sector is the A-image of the undeformed odd sector.

We now determine explicitly a basis of the invariant flag corresponding to the even sector of the first and second deformation. To do so, we switch to the algebraic variable and perform a change of gauge so that the undeformed, odd algebraic sector is isomorphic to the standard polynomial flag (cf case $\mathrm{III_o}$ of table 1).

$$\begin{aligned} T &= z(1-z)D_{zz} + \left(\left(\alpha-\frac{5}{2}\right)z+1-\alpha\right)D_z - \left(\frac{\alpha}{2}-\frac{3}{4}\right)^2, \\ T &= \mathrm{e}^{-\rho}H_{\mathrm{PT}}\,\mathrm{e}^{\rho}, \qquad \mathrm{e}^{\rho} = \sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)^{\frac{1}{2}-\alpha}, \\ z &= \cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}(\cosh x+1). \end{aligned} \tag{42}$$

In the algebraic gauge, the factorization functions for the state-adding transformations are given by

$$\phi = \mathrm{e}^{-\rho}\phi_{\mathrm{PT}}^{(m)} = z^{\alpha}(z-1)^{-\frac{1}{2}}P_m^{(-\frac{1}{2},\alpha)}(2z-1).$$

This factorization function is of algebraic type with

$$\begin{aligned} (\log\phi)_z &= \frac{\alpha}{z} + \frac{\frac{1}{2}}{1-z} + \left(\frac{1}{2}+\alpha+m\right)\frac{P_{m-1}^{(\frac{1}{2},\alpha+1)}(2z-1)}{P_m^{(-\frac{1}{2},\alpha)}(2z-1)} = \frac{a}{b}, \\ a &= \left(\left(\frac{1}{2}-\alpha\right)z+\alpha\right)P_m^{(-\frac{1}{2},\alpha)}(2z-1) + \left(\frac{1}{2}+\alpha+m\right)z(1-z)P_{m-1}^{(\frac{1}{2},\alpha+1)}(2z-1), \\ b &= z(1-z)P_m^{(-\frac{1}{2},\alpha)}(2z-1). \end{aligned}$$

A direct calculation shows that the above functions satisfy condition (28), and therefore $\hat{\phi} = 1$. The operators

$$A = bD_z - a, \qquad B = g^{-1}D_z,$$

where

$$g = P_m^{(-\frac{1}{2},\alpha)}(2z-1),$$

constitute a deformation pair of order m. Note that, as was shown in [3] and the references therein, the polynomial $g(z)$ does not vanish for $z \geqslant 1$.

Let us consider the cases $m = 1$ and $m = 2$ in more detail. For $m = 1$ we have

$$A = \frac{z_1(1-z_1)(z_1+2(\alpha+1))}{2(2\alpha+3)} D_{z_1} + \frac{(2\alpha+1)z_1^2 - 4(\alpha+1)}{4(2\alpha+3)},$$

$$B = \frac{2}{z_1} D_z,$$

$$\hat{T} = z(1-z)D_{zz} + \left(\frac{z_1}{2} - \frac{4(\alpha+1)}{(2\alpha+3)z_1} + \frac{2\alpha+1}{2\alpha+3}\right) D_z,$$

$$z_1 = 2P_1^{(-\frac{1}{2},\alpha)}(2z-1) = (2\alpha+3)z - 2\alpha - 2.$$

The operator $A = b(z_1)D_{z_1} - a(z_1)$, relative to the z_1 variable, is of the form (34), and hence, the partner operator $\hat{T}$ is P.E.S. with invariant subspaces

$$\hat{\mathcal{M}}_{n+1} = \langle 1, z_1^2, z_1^3, \ldots, z_1^n \rangle.$$

For $m = 2$ we set

$$z_2 = \tfrac{1}{2}\beta((2\alpha+7)z - (2\alpha+4)),$$

so that

$$P_2^{(-\frac{1}{2},\alpha)}(2z-1) = \frac{3(\alpha+2)}{2(2\alpha+7)}\left(z_2^2 - 1\right).$$

Consequently,

$$A = \left(\frac{-6\beta z_2^2}{2\alpha+5} - \frac{3(2\alpha+1)z_2}{3\alpha+6} + 3\beta\right) \frac{3(\alpha+2)^2\left(z_2^2-1\right)}{2(2\alpha+7)^2} D_{z_2}$$
$$+ \frac{3(\alpha+2)}{(2\alpha+7)^2}\left(-\frac{(2\alpha+3)}{2\beta}\left(z_2^3 - 3z_2\right) - 2\alpha - 1\right),$$

$$B = \frac{2(2\alpha+7)}{3(\alpha+2)\left(z_2^2-1\right)} D_z,$$

$$\hat{T} = z(1-z)D_{zz} + \left[\frac{3\beta(\alpha+2)}{2\alpha+5}\left(z_2 - \frac{4(2\alpha+3)}{2\alpha+7}\frac{z_2}{z_2^2-1}\right) + \frac{2(2\alpha+1)\left(z_2^2+1\right)}{(2\alpha+7)\left(z_2^2-1\right)}\right] D_z.$$

In the same manner, it can be seen from (35) that the partner operator $\hat{T}$ is P.E.S. with invariant subspaces (cf (36))

$$\hat{\mathcal{M}}_n = \langle 1, \pi_3(z_2), \pi_4(z_2), \ldots, \pi_{n+1}(z_2) \rangle.$$

4. Discussion

In this paper, we have analysed the connection between the Darboux transformations and exact solvability by polynomials, i.e., the fact that a certain Hamiltonian operator after a change of variables and a gauge transformation admits an infinite flag of invariant polynomial subspaces. Since operator composition and factorization are covariant with respect to changes of variables and gauge transformations, concepts like the Darboux transformation or shape invariance are also covariant. It is customary for physical applications to work in the Schrödinger gauge and the physical variable x, but for the purposes of analysing invariant flags it is more convenient to work in the algebraic variable z and in the algebraic gauge, the one in which the operator has polynomial eigenfunctions.

A general state-adding transformation on a shape-invariant potential will lead to a transformed potential whose eigenfunctions are no longer elementary functions. We have discussed the special class of *algebraic* Darboux transformations of shape-invariant potentials, i.e., those that preserve the exact solvability by polynomials, showing also how the polynomial flag is deformed by the action of the Darboux transformation.

In this paper, we placed our emphasis on the action of the Darboux transformation on the invariant flag of subspaces rather than on the potential. In fact, many different potentials have the same invariant flag, e.g., all the shape-invariant potentials preserve the standard polynomial flag. We have analysed the deformations of the Pöschl–Teller potential, but similar deformations exist for other shape-invariant forms.

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